

Noether Symmetries and Conservation Laws For Non-Critical Kohn-Laplace Equations on Three-Dimensional Heisenberg Group

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Abstract

We show which Lie point symmetries of non-critical semilinear Kohn-Laplace equations on the Heisenberg group H^1 are Noether symmetries and we establish their respective conservation laws.

1 Introduction and Main Results

In this paper we show which Lie point symmetries of the semilinear Kohn - Laplace equations on the three-dimensional Heisenberg group H^1 ,

$$\Delta_{H^1} u + f(u) = 0, \quad (1)$$

are Noether's symmetries, and we establish their respective conservation laws.

The Kohn - Laplace operator on H^1 is defined by

$$\Delta_{H^1} := X^2 + Y^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 4(x^2 + y^2) \frac{\partial^2}{\partial t^2} + 4y \frac{\partial^2}{\partial x \partial t} - 4x \frac{\partial^2}{\partial y \partial t},$$

where

$$X = \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} + 2x \frac{\partial}{\partial t}.$$

Eq.(1) possesses variational structure and can be derived from the Lagrangian

$$\mathcal{L} = \frac{1}{2} u_x^2 + \frac{1}{2} u_y^2 + 2(x^2 + y^2) u_t^2 + 2y u_x u_t - 2x u_y u_t - F(u), \quad (2)$$

with $F'(u) = f(u)$.

The group structure, the left invariant vector fields on H^1 and their Lie algebra are given, respectively, by $\phi : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$, where

$$\phi((x, y, t), (x_0, y_0, t_0)) := (x + x_0, y + y_0, t + t_0 + 2(xy_0 - yx_0)),$$

$$X = \frac{d}{ds} \phi((x, y, t), (s, 0, 0))|_{s=0} = \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial t},$$

$$Y = \frac{d}{ds} \phi((x, y, t), (0, s, 0))|_{s=0} = \frac{\partial}{\partial y} + 2x \frac{\partial}{\partial t}, \quad (3)$$

$$Z = \frac{d}{ds} \phi((x, y, t), (0, 0, s))|_{s=0} = \frac{\partial}{\partial t},$$

and

$$[X, T] = [Y, T] = 0, \quad [X, Y] = -4T.$$

In [2] a complete group classification for equation (1) is presented. It can be summarized as follows.

Let $G_f := \{T, R, \tilde{X}, \tilde{Y}\}$, where

$$T = \frac{\partial}{\partial t}, \quad R = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}, \quad \tilde{X} = \frac{\partial}{\partial x} - 2y \frac{\partial}{\partial t}, \quad \text{and} \quad \tilde{Y} = \frac{\partial}{\partial y} + 2x \frac{\partial}{\partial t}. \quad (4)$$

For any function $f(u)$, the group G_f is a (sub)group of symmetries. Its Lie algebra is summarized in Table 1.

	T	R	\tilde{X}	\tilde{Y}
T	0	0	0	0
R	0	0	\tilde{Y}	$-\tilde{X}$
\tilde{X}	0	$-\tilde{Y}$	0	4T
\tilde{Y}	0	\tilde{X}	-4T	0

Table 1: Lie brackets of equation (1) with $f(u)$ arbitrary.

For special choices of function $f(u)$ in (1), the symmetry group can be enlarged. Below we exhibit these functions and their respective additional symmetries and Lie algebras.

- If $f(u) = 0$, the additional symmetries are

$$\begin{aligned} V_1 = & (xt - x^2y - y^3) \frac{\partial}{\partial x} + (yt + x^3 + xy^2) \frac{\partial}{\partial y} \\ & + (t^2 - (x^2 + y^2)^2) \frac{\partial}{\partial t} - tu \frac{\partial}{\partial u}, \end{aligned} \quad (5)$$

$$\begin{aligned} V_2 = & (t - 4xy) \frac{\partial}{\partial x} + (3x^2 - y^2) \frac{\partial}{\partial y} \\ & - (2yt + 2x^3 + 2xy^2) \frac{\partial}{\partial t} + 2yu \frac{\partial}{\partial u}, \end{aligned} \quad (6)$$

$$\begin{aligned} V_3 = & (x^2 - 3y^2) \frac{\partial}{\partial x} + (t + 4xy) \frac{\partial}{\partial y} \\ & + (2xt - 2x^2y - 2y^3) \frac{\partial}{\partial t} - 2xu \frac{\partial}{\partial u}, \end{aligned} \quad (7)$$

$$Z = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 2t \frac{\partial}{\partial t},$$

$$U = u \frac{\partial}{\partial u}, \quad W_\beta = \beta(x, y, t) \frac{\partial}{\partial u}, \quad \text{where } \Delta_{H^1} \beta = 0. \quad (8)$$

	T	R	\tilde{X}	\tilde{Y}	U	W_β	V_1	V_2	V_3	Z
T	0	0	0	0	0	$W_{T\beta}$	V	\tilde{X}	\tilde{Y}	2T
R	0	0	\tilde{Y}	$-\tilde{X}$	0	$W_{R\beta}$	0	V_3	$-V_2$	0
\tilde{X}	0	$-\tilde{Y}$	0	4T	0	$W_{\tilde{X}\beta}$	V_2	-6R	2V	\tilde{X}
\tilde{Y}	0	\tilde{X}	$-4T$	0	0	$W_{\tilde{Y}\beta}$	V_3	-2V	-6R	\tilde{Y}
U	0	0	0	0	0	0	0	0	0	0
W_β	$-W_{T\beta}$	$-W_{R\beta}$	$-W_{\tilde{X}\beta}$	$-W_{\tilde{Y}\beta}$	0	0	$W_{V_1\beta}$	$W_{V_2\beta}$	$W_{V_3\beta}$	$W_{Z\beta}$
V_1	$-V$	0	$-V_2$	$-V_3$	0	$-W_{V_1\beta}$	0	0	0	$-2V_1$
V_2	$-\tilde{X}$	$-V_3$	6R	2V	0	$-W_{V_2\beta}$	0	0	4V ₁	$-V_2$
V_3	$-\tilde{Y}$	V_2	-2V	6R	0	$-W_{V_3\beta}$	0	-4V ₁	0	$-V_3$
Z	-2T	0	$-\tilde{X}$	$-\tilde{Y}$	0	$-W_{Z\beta}$	2V ₁	V_2	V_3	0

Table 2: Lie brackets of equation (1) with $f(u) = 0$. Here, $V := Z - U$.

- If $f(u) = u$, there are two additional symmetries, respectively, U and W_β as in Eq. (8), where $\Delta_{H^1} \beta + \beta = 0$.

	T	R	\tilde{X}	\tilde{Y}	U	W_β
T	0	0	0	0	0	$W_{T\beta}$
R	0	0	\tilde{Y}	$-\tilde{X}$	0	$W_{R\beta}$
\tilde{X}	0	$-\tilde{Y}$	0	4T	0	$W_{\tilde{X}\beta}$
\tilde{Y}	0	\tilde{X}	4T	0	0	$W_{\tilde{Y}\beta}$
U	0	0	0	0	0	0
W_β	$-W_{T\beta}$	$-W_{R\beta}$	$-W_{\tilde{X}\beta}$	$-W_{\tilde{Y}\beta}$	0	0

Table 3: Lie brackets of equation (1) with $f(u) = u$.

- If $f(u) = u^p$, $p \neq 0, p \neq 1, p \neq 3$, we have the generator of dilations

$$D_p = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 2t \frac{\partial}{\partial t} + \frac{2}{1-p} u \frac{\partial}{\partial u}. \quad (9)$$

	T	R	\tilde{X}	\tilde{Y}	D_p
T	0	0	0	0	2T
R	0	0	\tilde{Y}	$-\tilde{X}$	0
\tilde{X}	0	$-\tilde{Y}$	0	4T	\tilde{X}
\tilde{Y}	0	\tilde{X}	$-4T$	0	\tilde{Y}
D_p	-2T	0	$-\tilde{X}$	$-\tilde{Y}$	0

Table 4: Lie brackets of equation (1) with $f(u) = u^p$, $p \neq 0, p \neq 1, p \neq 3$.

- If $f(u) = e^u$ the additional symmetry is

$$E = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 2t \frac{\partial}{\partial t} - 2 \frac{\partial}{\partial u}. \quad (10)$$

	T	R	\tilde{X}	\tilde{Y}	E
T	0	0	0	0	2T
R	0	0	\tilde{Y}	$-\tilde{X}$	0
\tilde{X}	0	$-\tilde{Y}$	0	4T	\tilde{X}
\tilde{Y}	0	\tilde{X}	$-4T$	0	\tilde{Y}
E	-2T	0	$-\tilde{X}$	$-\tilde{Y}$	0

Table 5: Lie brackets of equation (1) with $f(u) = e^u$.

- In the critical case, $f(u) = u^3$, there are four additional generators, namely V_1, V_2, V_3 and D_3 , given in (5), (6), (7) and (9) respectively. Their Lie algebra is presented in [4].

In [3] is showed that in the critical case, $f(u) = u^3$, all Lie point symmetries are Noether symmetries and then, by the Noether Theorem (see [1], pag. 275), in [4] is established the respectives conservation laws for the symmetries $T, R, \tilde{X}, \tilde{Y}, V_1, V_2, V_3$ and D_3 .

In this work, we show which Lie point symmetries of the other functions $f(u)$ are Noether symmetries and then, we establish their respectives conservation laws, concluding the work started in [3] and [4].

Let $\mathbb{R} \ni u \mapsto F(u) \in \mathbb{R}$ be a differentiable function and

$$f(u) := F'(u). \quad (11)$$

Our main results can be formulated as follows:

Theorem 1 *The group G_f is a Noether symmetry group for any function $f(u)$ in (1).*

Theorem 2 *The Noether symmetry group of (1), with $f(u) = e^u$, is the group G_f .*

Theorem 3 *$G_f \cup \{W_\beta\}$ is the Noether symmetry group of equation (1), with $f(u) = u$ and β satisfies $\Delta_{H^1}\beta + \beta = 0$.*

Theorem 4 *The Noether symmetry group of equation (1) with $f(u) = 0$ is generated by the group G_f and by symmetries W_β, V_1, V_2 e V_3 , where β satisfies $\Delta_{H^1}\beta = 0$.*

As a consequence of theorems 1 - 4, we have the following conservation laws.

Theorem 5 *The conservations laws for the Noether symmetries of equation (1) for any $f(u)$ are:*

1. *For the symmetry T , the conservation law is $\text{Div}(\tau) = 0$, where $\tau = (\tau_1, \tau_2, \tau_3)$ and*

$$\begin{aligned} \tau_1 &= -2yu_t^2 - u_x u_t, \\ \tau_2 &= 2xu_t^2 - u_y u_t, \\ \tau_3 &= \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 - 2(x^2 + y^2)u_t^2 - F(u). \end{aligned}$$

2. For the symmetry R , the conservation law is $\text{Div}(\sigma) = 0$, where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and

$$\begin{aligned}\sigma_1 &= -\frac{1}{2}yu_x^2 + \frac{1}{2}yu_y^2 + 2y(x^2 + y^2)u_t^2 + xu_xu_y - yF(u), \\ \sigma_2 &= -\frac{1}{2}xu_x^2 - \frac{1}{2}xu_y^2 - 2x(x^2 + y^2)u_t^2 - yu_xu_y + xF(u), \\ \sigma_3 &= -2y^2u_x^2 - 2x^2u_y^2 + 4xyu_xu_y - 4y(x^2 + y^2)u_xu_t + 4x(x^2 + y^2)u_yu_t.\end{aligned}$$

3. For the symmetry \tilde{X} , the conservation law is $\text{Div}(\chi) = 0$, where $\chi = (\chi_1, \chi_2, \chi_3)$ and

$$\begin{aligned}\chi_1 &= -\frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + 2(x^2 + 3y^2)u_t^2 + 2yu_xu_t - 2xu_yu_t - F(u), \\ \chi_2 &= -4xyu_t^2 - u_xu_y + 2xu_xu_t + 2yu_yu_t, \\ \chi_3 &= -3yu_x^2 - yu_y^2 + 4y(x^2 + y^2)u_t^2 + 2xu_xu_y - 4(x^2 + y^2)u_xu_t + 2yF(u).\end{aligned}$$

4. For the symmetry \tilde{Y} , the conservation law is $\text{Div}(v) = 0$, where $v = (v_1, v_2, v_3)$ and

$$\begin{aligned}v_1 &= -4xyu_t^2 - u_xu_y - 2xu_xu_t - 2yu_yu_t, \\ v_2 &= \frac{1}{2}u_x^2 - \frac{1}{2}u_y^2 + 2(3x^2 + y^2)u_t^2 + 2yu_xu_t - 2xu_yu_t - F(u), \\ v_3 &= xu_x^2 + 3xu_y^2 - 4x(x^2 + y^2)u_t^2 - 2yu_xu_y - 4(x^2 + y^2)u_yu_t - 2xF(u).\end{aligned}$$

Theorem 6 *If $f(u) = 0$ in (1), the conservation laws for the Noether symmetries are as follows.*

1. For the symmetries T , R , \tilde{X} and \tilde{Y} , the conservation laws are the same as in the Theorem 5, with $f(u) = 0$, in (11).
2. For the symmetry V_1 , the conservation law is $\text{Div}(A) = 0$, where $A = (A_1, A_2, A_3)$ and

$$\begin{aligned}A_1 &= -\frac{1}{2}(tx - x^2y - y^3)u_x^2 + \frac{1}{2}(tx - x^2y - y^3)u_y^2 + 2t(x^3 + xy^2 - ty)u_t^2 \\ &\quad - (x^3 + xy^2 + ty)u_xu_y - [t^2 - (x^2 + y^2)^2]u_xu_t - 2t(x^2 + y^2)u_yu_t \\ &\quad - tuu_x - 2tyuu_t + yu^2,\end{aligned}$$

$$\begin{aligned}
A_2 = & \frac{1}{2}(x^3 + ty + xy^2)u_x^2 - \frac{1}{2}(x^3 + ty + xy^2)u_y^2 + 2t(x^2y + y^3 + tx)u_t^2 \\
& -(tx - x^2y - y^3)u_xu_y + 2t(x^2 + y^2)u_xu_t - [t^2 - (x^2 + y^2)^2]u_yu_t \\
& -tuu_y + 2txuu_t - xu^2, \\
A_3 = & \frac{1}{2}(t^2 - x^4 - 4txy + 2x^2y^2 + 3y^4)u_x^2 + \frac{1}{2}(t^2 + 3x^4 + 4txy + 2x^2y^2 - y^4)u_y^2 \\
& -2(x^2 + y^2)[t^2 - (x^2 + y^2)^2]u_t^2 + 2[t(x^2 - y^2) - 2xy(x^2 + y^2)]u_xu_y \\
& -4(x^2 + y^2)(tx - x^2y - y^3)u_xu_t - 4(x^2 + y^2)(x^3 + ty + xy^2)u_yu_t \\
& -2tyuu_x + 2txuu_y - 4t(x^2 + y^2)uu_t + 2(x^2 + y^2)u^2.
\end{aligned}$$

3. For the symmetry V_2 , the conservation law is $\text{Div}(B) = 0$, where $B = (B_1, B_2, B_3)$ and

$$\begin{aligned}
B_1 = & -\frac{1}{2}(t - 4xy)u_x^2 + \frac{1}{2}(t - 4xy)u_y^2 + [2t(x^2 + 3y^2) - 4xy(x^2 + y^2)]u_t^2 \\
& -(3x^2 - y^2)u_xu_y + 2(x^3 + ty + xy^2)u_xu_t - 2(tx - x^2y - y^3)u_yu_t \\
& + 2yuu_x + 4y^2uu_t, \\
B_2 = & \frac{1}{2}(3x^2 - y^2)u_x^2 - \frac{1}{2}(3x^2 - y^2)u_y^2 + 2(x^4 - 2txy - y^4)u_t^2 - (t - 4xy)u_xu_y \\
& + 2(tx - x^2y - y^3)u_xu_t + 2(x^3 + ty + xy^2)u_yu_t + 2yuu_y - 4xyuu_t - u^2, \\
B_3 = & (7xy^2 - x^3 - 3ty)u_x^2 + (5x^3 - 3xy^2 - ty)u_y^2 + 4(x^2 + y^2)(x^3 + ty + xy^2)u_t^2 \\
& + 2(tx - 7x^2y + y^3)u_xu_y - 4(t - 4xy)(x^2 + y^2)u_xu_t - 4(3x^4 + 2x^2y^2 - y^4)u_yu_t \\
& + 2xu^2 + 4y^2uu_x - 4xyuu_y + 8y(x^2 + y^2)uu_t.
\end{aligned}$$

4. For the symmetry V_3 , the conservation law is $\text{Div}(C) = 0$, where $C = (C_1, C_2, C_3)$ and

$$\begin{aligned}
C_1 = & -\frac{1}{2}(x^2 - 3y^2)u_x^2 + \frac{1}{2}(x^2 - 3y^2)u_y^2 + (2x^4 - 4txy - 2y^4)u_t^2 \\
& -(t + 4xy)u_xu_y + (2tx - 2x^2y + 2y^3)u_xu_t - (2x^3 + 2ty + 2xy^2)u_yu_t \\
& -4xyuu_t - 2xuu_x + u^2, \\
C_2 = & \frac{1}{2}(t + 4xy)u_x^2 - \frac{1}{2}(t + 4xy)u_y^2 + (6tx^2 + 4x^3y + 2ty^2 + 4xy^3)u_t^2 \\
& -(x^2 - 3y^2)u_xu_y + 2(x^3 + ty + xy^2)u_xu_t - 2(tx - x^2y - y^3)u_yu_t \\
& 2xu_yu + 4x^2u_tu, \\
C_3 = & (tx - 3x^2y + 5y^3)u_x^2 + (3tx + 7x^2y - y^3)u_y^2 \\
& (-4tx^3 + 4x^4y - 4txy^2 + 8x^2y^3 + y^5)u_t^2 + 2(x^3 - ty - 7xy^2)u_xu_y \\
& -2(2x^4 - 4x^2y^2 - 6y^4)u_xu_t - 4(x^2 + y^2)(t + 4xy)u_yu_t \\
& -8x^3uu_t - 8xy^2uu_t - 4x^2uu_y - 8xyuu_x + 2yu^2.
\end{aligned}$$

5. For the symmetry W_β , the conservation law is $\text{Div}(W) = 0$, where $W = (W_1, W_2, W_3)$ and

$$\begin{aligned}
W_1 &= \beta(u_x + 2yu_t) - u(\beta_x + 2y\beta_t), \\
W_2 &= \beta(u_y - 2xu_t) - u(\beta_y - 2x\beta_t), \\
W_3 &= \beta[-2xu_y + 2yu_x + 4(x^2 + y^2)u_t \\
&\quad + 2u[x\beta_y - y\beta_x - 2(x^2 + y^2)\beta_t]].
\end{aligned} \tag{12}$$

Theorem 7 *If $f(u) = u$ in (1), the conservation laws for the Noether symmetries are as follows.*

1. *For the symmetries T , R , \tilde{X} and \tilde{Y} , the conservation laws are the same as in the Theorem 5, with $f(u) = u$, in (11).*
2. *For the symmetry W_β , the conservation law is $\text{Div}(W) = 0$, where W is given in 12 .*

The remaining of the paper is organized as follows. In section 2 we briefly present some of the main aspects of Lie point symmetries, Noether symmetries and conservation laws. In section 3 we prove theorems 1, 2 and 3. Theorem 4 is proved in section 4. Their respective conservation laws are discussed in section 5.

2 Lie point symmetries, Noether symmetries and conservation laws

Let $x \in M \subseteq \mathbb{R}^n$, $u : M \rightarrow \mathbb{R}$ a smooth function and $k \in \mathbb{N}$. $\partial^k u$ denotes the jet bundle corresponding to all k th partial derivatives of u with respect to x . A *Lie point symmetry* of a partial differential equation (PDE) of order k , $F(x, u, \partial u, \dots, \partial^k u) = 0$, is a vector field

$$S = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta(x, u) \frac{\partial}{\partial u}$$

on $M \times \mathbb{R}$ such that $S^{(k)}F = 0$ when $F = 0$ and

$$S^{(k)} = S + \eta_i^{(1)}(x, u, \partial u) \frac{\partial}{\partial u_i} + \dots + \eta_{i_1 \dots i_k}^{(k)}(x, u, \partial u, \dots, \partial^k u) \frac{\partial}{\partial u_{i_1 \dots i_k}}$$

is the extended symmetry on the jet space $(x, u, \partial u, \dots, \partial^k u)$.

The functions $\eta^{(j)}(x, u, \partial u, \dots, \partial^j u)$, $1 \leq j \leq k$, are given by

$$\eta_i^{(1)} = D_i \eta - (D_i \xi^j) u_j,$$

$$\eta_{i_1 \dots i_j}^{(j)} = D_{i_j} \eta_{i_1 \dots i_{j-1}}^{(j-1)} - (D_{i_j} \xi^l) u_{i_1 \dots i_{j-1} l}, \quad 2 \leq j \leq k.$$

We are using the Einstein sum convention.

If the PDE $F = 0$ can be obtained by a Lagrangian $\mathcal{L} = \mathcal{L}(x, u, \partial u, \dots, \partial^l u)$ and if there exists some symmetry S of F and a vector $\varphi = (\varphi_1, \dots, \varphi_n)$ such that

$$S^{(l)} \mathcal{L} + \mathcal{L} D_i \xi^i = D_i \varphi^i, \quad (13)$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \dots + u_{ii_1 \dots i_m} \frac{\partial}{\partial u_{i_1 \dots i_m}} + \dots$$

is the total derivative operator of u ,

$$u_i := \frac{\partial u}{\partial x^i}, \quad u_{ij} := \frac{\partial^2 u}{\partial x^i \partial x^j}, \dots, u_{ii_1 \dots i_m} := \frac{\partial u}{\partial x_i \partial x_{i_1} \dots \partial x_{i_m}},$$

the symmetry S is said to be a *Noether symmetry*. Then, the Noether's Theorem asserts that the following conservation law holds

$$D_i (\xi^i \mathcal{L} + W^i [u, \eta - \xi^j u_j] - \varphi^i) = 0. \quad (14)$$

Above we have used the same notations and conventions as in [1]. (For the definition of W^i see [1], pp. 254-255.)

3 Proofs of theorems 1, 2 and 3

Lemma 8 *Let $u = u(x, y, t)$ be a smooth function. If a vector field $V = (A, B, C)$ is a vector function of $x, y, t, u, u_x, u_y, u_t$, its divergence necessarily depends on the second order derivatives of u with respect to x, y and t .*

Proof. Taking the divergence of vector field V , we obtain

$$\begin{aligned} \text{Div}(V) = & A_x + B_y + C_t + u_x A_u + u_{xx} A_{u_x} + u_{xy} A_{u_y} + u_{xt} A_{u_t} \\ & + u_y B_u + u_{xy} B_{u_x} + u_{yy} B_{u_y} + u_{yt} B_{u_t} \\ & + u_t C_u + u_{xt} C_{u_x} + u_{yt} C_{u_y} + u_{tt} C_{u_t}. \end{aligned}$$

■

Corollary 9 *If the divergence of a vector field does not depend on the second order derivatives, then it does not depend on u_x , u_y and u_t .*

Lemma 10 *The symmetry*

$$E = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 2t \frac{\partial}{\partial t} - 2 \frac{\partial}{\partial u}$$

is not a Noether symmetry.

Proof. In this case, $(\xi, \phi, \tau, \eta) = (x, y, 2t, -2)$. Then, $D_x \xi + D_y \phi + D_t \tau = 4$ and

$$(\eta_x^{(1)}, \eta_y^{(1)}, \eta_t^{(1)}) = (-u_x, -u_y, -2u_t),$$

which yields the following first order extension:

$$E^{(1)} = E - u_x \frac{\partial}{\partial u_x} - u_y \frac{\partial}{\partial u_y} - 2u_t \frac{\partial}{\partial u_t}.$$

Therefore,

$$\begin{aligned} E^{(1)} \mathcal{L} + (D_x \xi + D_y \phi + D_t \tau) \mathcal{L} &= u_x^2 + u_y^2 + 4(x^2 + y^2)u_t^2 \\ &\quad + 4yu_x u_t - 4xu_y u_t - 2e^u, \end{aligned} \tag{15}$$

where

$$\mathcal{L} := \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + 2(x^2 + y^2)u_t^2 + 2yu_x u_t - 2xu_y u_t - e^u.$$

From Lemma 8 and equation (15), we conclude that there are not a potential ϕ which satisfies

$$E^{(1)} \mathcal{L} + (D_x \xi + D_y \phi + D_t \tau) \mathcal{L} = \text{Div}(\phi).$$

Thus, E cannot be a Noether symmetry. ■

Lemma 11 *The symmetry U is not a Noether symmetry.*

Proof. First one, note that $\eta = u$, $\xi = \phi = \tau = 0$. Then,

$$U^{(1)} = u \frac{\partial}{\partial u} + u_x \frac{\partial}{\partial u_x} + u_y \frac{\partial}{\partial u_t} + u_t \frac{\partial}{\partial u_t} \tag{16}$$

Applying the operator obtained in (16) to the Lagrangian

$$\mathcal{L}_k := \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + 2(x^2 + y^2)u_t^2 + 2yu_xu_t - 2xu_yu_t - \frac{k}{2}u^2, \quad (17)$$

where $k = 0$ if $f(u) = 0$ or $k = 1$ if $f(u) = u$, we find

$$U^{(1)}\mathcal{L}_k = u_x^2 + u_y^2 + 4(x^2 + y^2)u_t^2 + 4yu_xu_t - 4xu_yu_t - ku^2 = 2\mathcal{L}_k.$$

From Lemma 8 and Corollary 9, we conclude that there is not a vector field such that equation (13) is true with $S = U$. ■

Lemma 12 *The symmetry W_β is a Noether symmetry.*

Proof. The first order extension $W^{(1)}$ of W is

$$W^{(1)} = \beta \frac{\partial}{\partial u} + \beta_x \frac{\partial}{\partial u_x} + \beta_y \frac{\partial}{\partial u_y} + \beta_t \frac{\partial}{\partial u_t}. \quad (18)$$

From (18) and (17), we have

$$W^{(1)}\mathcal{L}_k = -\beta ku + (u_x + 2yu_t)\beta_x + (u_y - 2xu_t)\beta_y + (4(x^2 + y^2)u_t + 2yu_x - 2xu_y)\beta_t$$

$$= \text{Div}((\beta_x + 2y\beta_t)u, (\beta_y - 2x\beta_t)u, (2y\beta_x - 2x\beta_y + 4(x^2 + y^2)\beta_t)u).$$

■

Lemma 13 *The symmetry*

$$Z = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 2t \frac{\partial}{\partial t}$$

is not a Noether symmetry.

Proof. Since $D_x\xi + D_y\phi + D_t\tau = 4$,

$$\mathcal{L} = \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + 2(x^2 + y^2)u_t^2 + 2yu_xu_t - 2xu_yu_t \quad (19)$$

and

$$Z^{(1)} = Z + u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + 2u_t \frac{\partial}{\partial t} \quad (20)$$

is a consequence of Eqs. (19) and (20), that

$$Z^{(1)}\mathcal{L} + \mathcal{L}(D_x\xi + D_y\phi + D_t\tau) = 2\mathcal{L}. \quad (21)$$

By Lemma 8, there does not exist a vector field such that the right hand of (21) be its divergence. ■ **Proof of Theorem 1:** We will use four steps to prove this theorem. First, we obtain the first order extension of symmetries T , R , \tilde{X} , \tilde{Y} . Next, we proof the theorem for each one of them.

1. Extensions:

- (a) Symmetry T The coefficients of T are $\xi = \phi = \eta = 0$ and $\phi = 1$. Then

$$T^{(1)} = T.$$

- (b) Symmetry R The coefficients of symmetry R are $(\xi, \phi, \tau, \eta) = (y, -x, 0, 0)$. Then, we conclude that

$$R^{(1)} = R + u_y \frac{\partial}{\partial u_x} - u_x \frac{\partial}{\partial u_y}.$$

- (c) Symmetry \tilde{X} In this case, $(\xi, \phi, \tau, \eta) = (1, 0, -2y, 0)$. Then

$$\eta_x^{(1)} = 0, \quad \eta_y^{(1)} = 2u_t, \quad \eta_t^{(1)} = 0$$

and

$$\tilde{X}^{(1)} = \tilde{X} + 2u_t \frac{\partial}{\partial u_y}.$$

- (d) Symmetry \tilde{Y} This case is analogous to case c and we present only its extension

$$\tilde{Y}^{(1)} = \tilde{Y} - 2u_t \frac{\partial}{\partial u_x}.$$

Corollary 14 *The divergence of any symmetry $S \in G_f$ is zero.*

2. (a) Proof of theorem for the symmetry T . Since $Div(T) = 0 = T^{(1)}\mathcal{L}$ it is immediate that

$$T^{(1)}\mathcal{L} + \mathcal{L}Div(T) = 0.$$

(b) Proof of theorem for the symmetry R . We have

$$R^{(1)}\mathcal{L} = 0.$$

Then, from Corollary 14,

$$R^{(1)}\mathcal{L} + \mathcal{L}Div(R) = 0.$$

(c) Proof of theorem for the symmetries \tilde{X} and \tilde{Y} . It is immediate that

$$\tilde{X}^{(1)}\mathcal{L} = 0.$$

Again, by Corollary 14, we obtain

$$\tilde{X}^{(1)}\mathcal{L} + \mathcal{L}Div(\tilde{X}) = 0.$$

In the same way, we conclude that

$$\tilde{Y}^{(1)}\mathcal{L} + \mathcal{L}Div(\tilde{Y}) = 0.$$

Proof of Theorem 2: It is a consequence of Lemma 10 and Theorem 1.

Proof of Theorem 3: From Lemma 11, U is not a Noether symmetry. Then, by Theorem 1 and Lemma 12, $G_f \cup \{W_\beta\}$ is a Noether symmetry group.

Proof of Theorem 4: By lemmas 11 and 13, the symmetries Z and U are not Noether symmetries. The proof that the symmetries V_1 , V_2 and V_3 are Noether symmetries is obtained in same way that Bozhkov and Freire showed that V_1 , V_2 and V_3 are Noether symmetries of (1) when $f(u) = u^3$, and can be found in [3]. Then, by Theorem 1 and Lemma 12, we conclude the proof.

4 Conservation Laws

The proof is by a straightforward calculation, which we shall not present here. However, a computer assisted proof can be obtained by means of the software *Mathematica*. It calculates the components of the conservation laws, which appear in the equation (14). The Mathematica notebook used for this purpose can be obtained from the author upon request.

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